MODULE HOMOMORPHISMS OF A VON NEUMANN ALGEBRA INTO ITS CENTER[1]

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1. **Introduction.** A tool for studying von Neumann algebras is the space of σ-weakly continuous linear functionals. There are certain limitations in the applicability of this tool because the center of the algebra is somehow submerged and on the whole it seems σ-weakly continuous functionals are more suitable for studying factor algebras. In this article we propose another tool, viz. the σ-weakly continuous module homomorphisms of the von Neumann algebra $\mathscr{A}$, considered as a module over its center $\mathscr{Z}$, into $\mathscr{Z}$. First it is shown that every σ-weakly continuous linear functional can be written as the composition of a σ-weakly continuous module homomorphism into $\mathscr{Z}$ and a σ-weakly continuous linear functional on $\mathscr{Z}$. For these module homomorphisms there is obtained a specific form which generalizes the well-known form of σ-weakly continuous linear functionals ([1], [16]). Certain facts concerning decomposition theory result. Further $\mathscr{A}$ is the dual (in terms of module homomorphisms) of the space of all σ-weakly continuous module homomorphisms of $\mathscr{A}$ into $\mathscr{Z}$. As an example of the applicability of this tool a type I algebra is described as the second dual (in terms of module homomorphisms) of a module. Further applications are contained in a later paper.

It seems reasonable to conjecture from the ensuing proofs that considering the module structure might further illuminate the relation between a von Neumann algebra and an AW^*-algebra.

2. **Extension of homomorphisms.** Let $\mathscr{Z}$ be a commutative von Neumann algebra. An algebraic $\mathscr{Z}$-module $\mathscr{M}$ with norm $\|\cdot\|$ will be called a Banach $\mathscr{Z}$-module if (1) $\mathscr{M}$ is a Banach space, and (2) $\|CB\| \leq \|C\| \, \|B\|$ for every $C \in \mathscr{Z}$ and $B \in \mathscr{M}$. A module homomorphism of $\mathscr{M}$ into $\mathscr{Z}$ will be called a functional and the set of all bounded functionals of the module $\mathscr{M}$ will be denoted by $\mathscr{M}^{\sim}$. The space $\mathscr{M}^{\sim}$ with the operator bound is again a Banach $\mathscr{Z}$-module. If $\mathscr{A}$ is a von Neumann algebra with center $\mathscr{Z}$, then $\mathscr{A}$ is a Banach $\mathscr{Z}$-module. Whenever we talk about the module $\mathscr{A}$, we shall have this particular module in mind. A functional ϕ of the module $\mathscr{A}$ is said to be hermitian (resp. positive) if $\phi(A)$ is hermitian (resp. positive) in $\mathscr{Z}$ whenever A is hermitian (resp. positive) in $\mathscr{A}$. A positive functional ϕ of the algebra (resp. module) $\mathscr{A}$ is said to be normal if lub $\phi(A_n)=\phi(A)$ for any monotonely increasing net $\{A_n\}$ of hermitian elements of $\mathscr{A}$ with least upper bound A.

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Then the positive functional ϕ is a normal functional of the algebra (respectively, module) $\mathscr{A}$ if and only if ϕ (resp. the composition $f\cdot\phi$) is an element of the associated space $\mathscr{A}_*$ of $\mathscr{A}$ (resp. for every f in the associated space $\mathscr{Z}_*$ of $\mathscr{Z}$) [2, I, §4, Theorem 1]. If E is an abelian projection with central support P of the commutator $\mathscr{Z}'$ of $\mathscr{Z}$, then $CP\to CE$ is an isomorphism of $\mathscr{Z}P$ onto $E\mathscr{Z}'E$. For each A in $\mathscr{Z}'$ let $\tau_E(A)$ denote the inverse image of EAE under this isomorphism. The map τ_E restricted to $\mathscr{A}$ is then easily seen to be a normal functional of the module $\mathscr{A}$.

The following theorem shows the relation between functionals of $\mathscr{A}$ and the module $\mathscr{A}$.

THEOREM 1. *Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{Z}$ and let f be a normal functional on $\mathscr{A}$. Then there is a normal functional ϕ of the module $\mathscr{A}$ with $\phi(1)$ equal to the support P of f restricted to $\mathscr{Z}$ such that $f=f\cdot\phi$. The functional ϕ is unique in the following sense: if ψ is a functional of the module $\mathscr{A}$ with $f=f\cdot\psi$, then $P\psi(A)=\phi(A)$ for every A in $\mathscr{A}$.*

Proof. We may assume $f\neq 0$. Let Φ be the canonical representation of $\mathscr{A}$ on a Hilbert space H induced by f and let x be a cyclic vector under $\Phi(\mathscr{A})$ in H such that $f(A)=w_x\cdot\Phi(A)=(\Phi(A)x,x)$ for every A in $\mathscr{A}$ [2, I, §4]. There is a central projection Q such that $\mathscr{A}(1-Q)$ is the kernel of Φ and such that Φ carries $\mathscr{A}Q$ isomorphically onto $\Phi(\mathscr{A})$. We observe that $1-P$ is in the kernel of Φ and so $Q\leqq P$. However, $f(P-Q)=w_x(\Phi(P-Q))=0$; therefore $P=Q$. Now the inverse θ of Φ restricted to $\mathscr{A}P$ is a σ-weakly continuous homomorphism of $\Phi(\mathscr{A})$ onto $\mathscr{A}P$ [2, I, §4, Corollaries, Theorem 2]. Let E be the abelian projection of the commutator $\Phi(\mathscr{Z})'$ of $\Phi(\mathscr{Z})$ on H corresponding to the subspace closure $\{Ax \mid A\in\Phi(\mathscr{Z})\}$ [2, I, §2, no. 1, and §6, Proposition 4, Corollary 2]. Let $\tau=\tau_E$. Then we can easily verify that the functional $\phi=\theta\cdot\tau\cdot\Phi$ of the module $\mathscr{A}$ into $\mathscr{Z}P$ is normal, and that $f(A)=f(\Phi(A))$.

Now let ψ be a functional of the module $\mathscr{A}$ such that $f\cdot\psi=f$. There is a vector y in the Hilbert space of $\mathscr{A}$ such that $f=w_y$ on $\mathscr{Z}$ [2, III, §1, Corollary to Theorem 4]. We have that $(\psi(A)By, Cy)=(\phi(A)By, Cy)$ for every B and C in $\mathscr{Z}$ and every A in $\mathscr{A}$. If F is the projection corresponding to closure $\{By \mid B\in\mathscr{Z}\}$, then $\psi(A)F=\phi(A)F$ for every A in $\mathscr{A}$. Because the central support of F is P, we have that $\psi(A)P=\phi(A)$ for every A in $\mathscr{A}$. Q.E.D.

We now determine the form of a normal functional in terms of the trace class (cf. [9, §4]).

THEOREM 2. *Let $\mathscr{A}$ be a von Neumann algebra on a Hilbert space H. If ϕ is a normal functional of the module $\mathscr{A}$, then there is a positive element B in the trace class of the commutator $\mathscr{Z}'$ of the center $\mathscr{Z}$ of $\mathscr{A}$ such that $\phi(A)=\mathrm{Tr}\,(AB)$.*

Proof. There is a set $\{P_i \mid i\in S\}$ of mutually orthogonal projections of $\mathscr{Z}$ of sum 1 such that each algebra $\mathscr{Z}P_i$ on P_iH has a separating unit vector x_i [2, I, §2]. Since the functional $w_{x_i}\cdot\phi=f_i$ restricted to $\mathscr{A}P_i$ is normal, there is a normal functional g_i of the commutator $\mathscr{Z}'P_i$ of $\mathscr{Z}P_i$ on P_iH that coincides with f_i on $\mathscr{A}P_i$. However, by

Theorem 1 there is a normal functional ψ_i of the module $\mathscr{Z}'P_i$ such that $\psi_i(P_i) \leqq P_i$ and $g_i \cdot \psi_i = g_i$. There is a y_i in P_iH such that $g_i(A) = (Ay_i, y_i)$ for every A in $\mathscr{Z}P_i$. Then $(Ay_i, y_i) = f_i(A) = (A\phi(1)x_i, x_i)$ for every A in $\mathscr{Z}P_i$. So $(\phi(1)\psi_i(A)Bx_i, Cx_i) = (\phi(A)Bx_i, Cx_i)$ for every A in $\mathscr{A}P_i$ and B and C in $\mathscr{Z}P_i$. Because x_i is separating for $\mathscr{Z}P_i$ we see that $\phi(A) = \phi(1)\psi_i(A)$ for every A in $\mathscr{A}P_i$. There is a unique positive element B_i in the trace class of $\mathscr{Z}'P_i$ such that $\phi(1)\psi_i(A) = \mathrm{Tr}\,(AB_i)$ for every A in $\mathscr{Z}'P_i$ [9, Theorem 4.13]. But for every finite subset π of S we have that $\mathrm{Tr}\,(\sum_\pi B_i) \leqq \phi(1)$. So $B = \sum B_i$ is a positive element of the trace class of $\mathscr{Z}'$ such that $\mathrm{Tr}\,(AB) = \phi(A)$ for every A in $\mathscr{A}$ since $\mathrm{Tr}\,(AB)P_i = \mathrm{Tr}\,((AP_i)B_i) = \phi(AP_i) = \phi(A)P_i$ for every P_i. Q.E.D.

COROLLARY. *Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{Z}$. If ϕ is a normal functional of the module $\mathscr{A}$, there is a monotone decreasing sequence $\{B_n\}$ of elements in $\mathscr{Z}^+$ and an orthogonal sequence $\{E_n\}$ of abelian projections of the commutator $\mathscr{Z}'$ of $\mathscr{Z}$ such that* (1) $\lim B_n = 0$ (*uniformly*); (2) $\sum B_n$ *is bounded above*; *and* (3) $\phi(A) = \sum B_n \tau_{E_n}(A)$ *for every* $A \in \mathscr{A}$.

Proof. This results from [9, §2 and §4].

REMARK. Theorem 2 gives rise to a representation of a normal functional f on a von Neumann algebra $\mathscr{A}$ as an integral over the spectrum Z of the center $\mathscr{Z}$ of $\mathscr{A}$. Let ν be the so-called spectral measure on Z induced by the relation $f(A) = \int_Z A^\wedge(\zeta)\, d\nu(\zeta)$ $(A \in \mathscr{Z})$. Here $A^\wedge$ denotes the Gelfand transform of A in $\mathscr{Z}$. There is a positive element B in the trace class of the commutator of $\mathscr{Z}$ such that $f(A) = f(\mathrm{Tr}\,(AB)) = \int \mathrm{Tr}\,(AB)^\wedge(\zeta)\, d\nu(\zeta)$. In the same notation as the preceding corollary let $f_{n\zeta}(A) = (B_n\tau_{E_n}(A))^\wedge(\zeta)$ for every $\zeta \in Z$. Then $f_{n\zeta}$ is a positive functional on $\mathscr{A}$ whose kernel contains the ideal ζ in Z. If $\mathscr{A}$ is discrete, the nonzero $f_{n\zeta}$ are irreducible (cf. [8, Proposition 2.1]). If $\mathscr{A}$ is continuous, then $f_{n\zeta}(1)^{-1}f_{n\zeta}$ is in the pure state space of $\mathscr{A}$ whenever $f_{n\zeta} \neq 0$ [4, Theorem 3]. Now for fixed $A \in \mathscr{A}^+$ $\{\sum \{B_n\tau_{E_n}(A) \mid 1 \leqq n \leqq m\}\}_m$ is a monotonely increasing sequence which converges strongly to $\mathrm{Tr}\,(AB)$. There is a subset N of ν-measure 0 such that

$$\lim_m \sum \{f_{n\zeta}(A) \mid 1 \leqq n \leqq m\} = \mathrm{Tr}\,(BA)^\wedge(\zeta)$$

for every $\zeta \notin N$. The restriction that $A \in \mathscr{A}^+$ can certainly be removed and so, we obtain the formula $\sum_n \int f_{n\zeta}(A)\, d\nu(\zeta) = \int \mathrm{Tr}\,(BA)^\wedge(\zeta)\, d\nu(\zeta) = f(A)$ for every A in $\mathscr{A}$ by the dominated convergence theorem.

We state some additional facts for type I algebras. Here we may assume $\mathscr{A} = \mathscr{Z}'$. The support X of ν is open and closed and corresponds to a projection P in $\mathscr{Z}$ under the relation $\{\zeta \in Z \mid P^\wedge(\zeta) = 1\} = X$. Let E be an abelian projection of central support P. If $\zeta \in X$ and if $[\zeta]$ denotes the smallest closed two-sided ideal in $\mathscr{A}$ containing ζ, then the factor space $\mathscr{A}E/[\zeta]$ becomes a Hilbert space $H(\zeta)$ with inner product $(A(\zeta), B(\zeta)) = \tau_E(B^*A)^\wedge(\zeta)$. Here $A(\zeta)$ denotes the image of A in $\mathscr{A}/[\zeta]$. Let Φ_ζ be the representation by left multiplication of $\mathscr{A}$ with kernel $[\zeta]$ on $H(\zeta)$. There is a sequence $\{P_k\}$ of mutually orthogonal projections in $\mathscr{Z}$ of sum P such that $\{\sum_n B_n P_k\}$ converges uniformly for each P_k (cf. [9, Theorem 4.1]). Let $X_k=\{\zeta \in Z \mid P_k^{\frown}(\zeta)=1\}$. For each $\zeta \in \bigcup X_k$ there is a normal functional f_ζ on the algebra L of bounded linear operators of $H(\zeta)$ with $f_\zeta \cdot \Phi_\zeta(A)=\text{Tr}\,(BA)^{\frown}(\zeta)$. Indeed, there is a normal functional $g_{n\zeta}$ on L with $g_{n\zeta}\cdot \Phi_\zeta = f_{n\zeta}$. Since the set $N=X-\bigcup X_k$ has ν-measure 0, we have that $f(A)=\int f_\zeta(A(\zeta))\, d\nu(\zeta)$, [9, §4].

Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{Z}$. The set $\mathscr{A}_\sim$ of all functionals of the module $\mathscr{A}$ which are continuous in the $\sigma(\mathscr{A}, \mathscr{A}_*)$ and $\sigma(\mathscr{Z}, \mathscr{Z}_*)$ topologies of $\mathscr{A}$ and $\mathscr{Z}$ respectively is an algebraic $\mathscr{Z}$-module. The module $\mathscr{A}_\sim$ is a closed subspace of the space $\mathscr{A}^\sim$ of all bounded functionals of the module $\mathscr{A}$ because the uniform limit of a sequence of σ-weakly continuous functionals of $\mathscr{A}$ is σ-weakly continuous. Thus $\mathscr{A}_\sim$ is a Banach $\mathscr{Z}$-module.

In passing we remark that analogues of theorems for functionals of $\mathscr{A}_*$ may be formulated and proved for functionals of $\mathscr{A}_\sim$ using Theorem 1. In particular, we may obtain a polar decomposition for functionals of $\mathscr{A}_\sim$ [14] as well as a decomposition of functionals of $\mathscr{A}_\sim$ into a linear combination of normal functionals. The proof of the latter statement is virtually identical with that of Theorem 4.12 of [9]. In particular, if ϕ is a hermitian functional of $\mathscr{A}_\sim$, then there is a projection $E \in \mathscr{A}$ such that $\phi_1(A)=\phi(AE)$ and $\phi_2(A)=-\phi(A(1-E))$ are normal functionals of the module $\mathscr{A}$. Also a Radon-Nikodym theorem [15] holds.

The next theorem is an analogue of a well-known result concerning linear functionals on a von Neumann algebra. The theorem though is potentially applicable to the theory of AW^*-algebras in a way precluded for linear functionals (cf. [13]) and so is interesting in its own right. We plan to explore this application in a future paper.

THEOREM 3. *Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{Z}$. For each A in $\mathscr{A}$ define the function F_A on $\mathscr{A}_\sim$ by $F_A(\phi)=\phi(A)$. Then the function $A \to F_A$ defines an isometric isomorphism of the module $\mathscr{A}$ onto the module $(\mathscr{A}_\sim)^\sim$.*

Proof. First it is easy to see that $F_A \in (\mathscr{A}_\sim)^\sim$ for each $A \in \mathscr{A}$ and that the function $A \to F_A$ is a module homomorphism of $\mathscr{A}$ into $(\mathscr{A}_\sim)^\sim$. Now let $\mathscr{Z}'$ be the commutator of $\mathscr{Z}$; let $\varepsilon>0$ be given. There is an abelian projection E in $\mathscr{Z}'$ such that $\|E|A|E\| \geqq \|A\|-\varepsilon$ where $|A|=(A^*A)^{1/2}$. Let U be a partial isometry of $\mathscr{A}$ such that $A=U|A|$ is the polar decomposition of A. Defining $\phi(B)=\tau_E(U^*B)$ for $B \in \mathscr{A}$, we see that $\|\phi(B)\|^2 \leqq \|\tau_E(U^*U)\|\,\|\tau_E(B^*B)\| \leqq \|B\|^2$ for every $B \in \mathscr{A}$. Thus ϕ is in the unit sphere of $\mathscr{A}_\sim$. However $\|F_A(\phi)\| \geqq \|A\|-\varepsilon$. Because $\varepsilon>0$ is arbitrary $\|F_A\| \geqq \|A\|$. Since it is obviously true that $\|F_A\| \leqq \|A\|$, we have that $\|F_A\|=\|A\|$. Hence the map $A \to F_A$ is an isometry.

We complete the proof by showing $A \to F_A$ is onto $(\mathscr{A}_\sim)^\sim$. Let f be an element of $(\mathscr{A}_\sim)^\sim$. For each ϕ in $\mathscr{Z}'_\sim$ let $F(\phi)$ denote the restriction of ϕ to $\mathscr{A}$. Then F is a continuous homomorphism of the $\mathscr{Z}$-module $\mathscr{Z}'_\sim$ onto the $\mathscr{Z}$-module $\mathscr{A}_\sim$. This means that $f\cdot F$ is an element of $(\mathscr{Z}'_\sim)^\sim$. There is an element A in $\mathscr{Z}'$ such that

$f \cdot F(\phi) = \phi(A)$ for every ϕ in $\mathscr{Z}'_{\sim}$ [9, Theorems 4.10 and 4.13]. We show that $A \in \mathscr{A}$. Indeed, if $A \notin \mathscr{A}$ then there is a unitary operator V in the commutator $\mathscr{A}'$ of $\mathscr{A}$ such that $VAV^* \neq A$. There is an abelian projection G in $\mathscr{Z}'$ such that $GVAV^*G \neq GAG$. Let G' be the abelian projection in $\mathscr{Z}'$ given by $G' = V^*GV$. If $\phi = \tau_{G'} - \tau_G$, then $\phi(B) = 0$ for every B in $\mathscr{A}$; but $\phi(A) = \tau_G(VAV^* - A) \neq 0$. Thus $0 = f \cdot F(\phi) = \phi(A) \neq 0$. This is impossible. So we conclude that $A \in \mathscr{A}$. Now by the preliminary remarks concerning the decomposition of elements of $\mathscr{A}_{\sim}$ into a linear combination of normal functionals and by Theorem 2 we find that the range of F is $\mathscr{A}_{\sim}$. If $\phi \in \mathscr{A}_{\sim}$, then there is a $\psi \in \mathscr{Z}'_{\sim}$ such that $F(\psi) = \phi$ and thus $f(\phi) = \psi(A) = \phi(A)$ since $A \in \mathscr{A}$. This completes the proof. Q.E.D.

3. **Characterization of type I algebras.** It is known that a C^*-algebra that can be expressed as the second dual of a Banach space is a von Neumann algebra with minimal projections ([13], [7], [19]). So linear functionals are suitable for characterizing type I factor algebras but unsuitable for characterizing algebras with arbitrary centers. The object of this section is to bring the center into consideration by considering module homomorphisms and thereby to obtain a description of type I algebras.

THEOREM 4. *Let $\mathscr{Z}$ be a commutative von Neumann algebra and let $\mathscr{A}$ be a C^*-algebra with center $\mathscr{Z}$. Let $\mathscr{M}$ be a Banach $\mathscr{Z}$-module. If $\mathscr{A}$ is the dual $\mathscr{M}^{\sim}$ of $\mathscr{M}$, then $\mathscr{A}$ is a von Neumann algebra and every functional $A \to \phi(A)$ of $\mathscr{A}$ into $\mathscr{Z}$ for fixed ϕ in $\mathscr{M}$ is a σ-weakly continuous functional of the module $\mathscr{A}$.*

NOTE. We write $\phi(A)$ to indicate the functional $A \in \mathscr{A}$ applied to $\phi \in \mathscr{M}$.

Proof. Let $\mathscr{B}$ be the linear manifold in the dual $\mathscr{A}^*$ of $\mathscr{A}$ generated by the set $\{f \cdot \phi \mid f \in \mathscr{Z}_*, \phi \in \mathscr{M}\}$. We show that the dual $\mathscr{B}^*$ of $\mathscr{B}$ is identified with $\mathscr{A}$. Then by Sakai's Theorem [13, Theorem 1 and Remark 1] $\mathscr{A}$ is a von Neumann algebra and the functionals of $\mathscr{B}$ are σ-weakly continuous on $\mathscr{A}$. In fact the closure of $\mathscr{B}$ in $\mathscr{A}^*$ is equal to the associated space of $\mathscr{A}$ [12, 22.6]. This means that the elements of $\mathscr{M}$ are σ-weakly continuous. We proceed to identify $\mathscr{B}^*$ with $\mathscr{A}$. Let Φ be an element of $\mathscr{B}^*$. Let H be a Hilbert space on which $\mathscr{Z}$ may be represented as a von Neumann algebra. The functional $w_{x,y} \cdot \phi$ is in $\mathscr{B}$ for every x and y in H and $\phi \in \mathscr{M}$. Here $w_{x,y}(A) = (Ax, y)$. Setting $\langle x, y \rangle_\phi = \Phi(w_{x,y} \cdot \phi)$ we obtain for each $\phi \in \mathscr{M}$ a bilinear form on H. Since $|\langle x, y \rangle_\phi| \leqq \|x\| \, \|y\| \, \|\phi\| \, \|\Phi\|$, the form is continuous and hence there is a unique bounded linear operator A_ϕ on H such that $\langle x, y \rangle_\phi = (A_\phi x, y)$ for every x and y in H. If B is an arbitrary element in the commutator $\mathscr{Z}'$ of $\mathscr{Z}$ on H, then $(A_\phi Bx, y) = (BA_\phi x, y)$ for every x and y in H because $w_{Bx,y} \cdot \phi = w_{x,B^*y} \cdot \phi$. This means A_ϕ is in the double commutator of $\mathscr{Z}$ which is $\mathscr{Z}$. The map $\phi \to A_\phi$ is easily seen to be a module homomorphism of $\mathscr{M}$ into $\mathscr{Z}$. It is bounded because

$$\|A_\phi\| = \text{lub}\,\{|\Phi(w_{x,y} \cdot \phi)| \mid \|x\| \leqq 1, \|y\| \leqq 1\} \leqq \|\Phi\| \, \|\phi\|.$$

By hypothesis there is a unique A_0 in $\mathscr{A}$ such that $\phi(A_0) = A_\phi$ for every $\phi \in \mathscr{M}$. Then for every $g \in \mathscr{B}$ we have that $\Phi(g) = g(A_0)$.

For each $A \in \mathscr{A}$ let G_A be the element of $\mathscr{B}^*$ defined by $G_A(g)=g(A)$. The map $A \to G_A$ of $\mathscr{A}$ into $\mathscr{B}^*$ is a linear function of $\mathscr{A}$ whose range is $\mathscr{B}^*$ by the preceding paragraph. We complete the proof by showing that this map is an isometry. Because $\mathscr{B}$ is a subset of $\mathscr{A}^*$ we have that $\|G_A\| \leqq \|A\|$. On the other hand the two relations

$$\text{lub}\,\{\|\phi(A)\| \mid \phi \in \mathscr{M},\ \|\phi\| \leqq 1\} = \|A\|$$

and

$$\text{lub}\,\{|f\cdot\phi(A)| \mid f \in \mathscr{Z}_*,\ \|f\| \leqq 1\} = \|\phi(A)\|$$

imply the relation $\|A\| \leqq \|G_A\|$. So $\|A\| = \|G_A\|$. Q.E.D.

Before proceeding we make some observations. Assume that $\mathscr{M}$ is a Banach modulo over a commutative von Neumann algebra $\mathscr{Z}$ whose second dual $\mathscr{M}^{\sim\sim}$ is a C^*-algebra $\mathscr{A}$ with center $\mathscr{Z}$. Let $\mathscr{M}'$ be the Banach $\mathscr{Z}$-module $\mathscr{M}$ reduced module $\mathscr{K} = \bigcap \{\text{kernel}\,\phi \mid \phi \in \mathscr{M}^{\sim}\}$ with the norm $\|A'\| = \text{glb}\,\{\|A+B\| \mid B \in \mathscr{K}\}$. Here A' is the image in $\mathscr{M}'$ of A in $\mathscr{M}$ under the canonical homomorphism. It is standard to show that $\mathscr{M}^{\sim}$ is identified with (i.e. is isometric isomorphic to) the dual $\mathscr{M}'^{\sim}$ under the correspondence $\phi \to \phi'$ where $\phi'(A') = \phi(A)$. By replacing $\mathscr{M}$ by $\mathscr{M}'$ if necessary we may assume that $\mathscr{K} = (0)$. For each $A \in \mathscr{M}$ the function $\phi \to \phi(A)$ defines an element F_A of $\mathscr{M}^{\sim\sim} = \mathscr{A}$. The map $A \to F_A$ is an isomorphism of $\mathscr{M}$ into a submodule $\mathscr{N}$ of the module $\mathscr{A}$. Setting $\phi'(A) = \phi(F_A)$ for each $\phi \in \mathscr{N}^{\sim}$ and $A \in \mathscr{M}$, we obtain a functional ϕ' of $\mathscr{M}^{\sim}$ because $\|F_A\| \leqq \|A\|$. It satisfies the relation

$$\begin{aligned}\|\phi\| &= \text{lub}\,\{\|\phi(A)\| \mid A \in \mathscr{N},\ \|A\| \leqq 1\} \\ &\geqq \text{lub}\,\{\|\phi'(A)\| \mid A \in \mathscr{M},\ \|A\| \leqq 1\} = \|\phi'\| \\ &\geqq \text{lub}\,\{\|\phi'(A)\| \mid A \in \mathscr{A},\ \|A\| \leqq 1\} \geqq \|\phi\|.\end{aligned}$$

This means that $\phi \to \phi'$ is an isometric isomorphism of $\mathscr{N}^{\sim}$ onto $\mathscr{M}^{\sim}$. Because it is well known that the dual of the closure of $\mathscr{N}$ is equal to $\mathscr{N}^{\sim}$, we may assume that $\mathscr{M}$ is embedded in $\mathscr{A}$. For every A in $\mathscr{M}$ we have that

$$\|A\| = \text{lub}\,\{\|\phi(A)\| \mid \phi \in \mathscr{M}^{\sim},\ \|\phi\| \leqq 1\} \tag{1}$$

because $\mathscr{M}^{\sim\sim} = \mathscr{A}$ and for every $\phi \in \mathscr{M}^{\sim}$ we have that

$$\begin{aligned}\|\phi\| &= \text{lub}\,\{\|\phi(A)\| \mid A \in \mathscr{M},\ \|A\| \leqq 1\} \\ &\leqq \text{lub}\,\{\|\phi(A)\| \mid A \in \mathscr{A},\ \|A\| \leqq 1\} \leqq \|\phi\|.\end{aligned} \tag{2}$$

So we are always justified in assuming that $\mathscr{M}$ is embedded in $\mathscr{A}$ and that (1) and (2) hold.

Theorem 5. *Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{Z}$. If $\mathscr{M}$ is a closed submodule of the module $\mathscr{A}$ whose second dual $\mathscr{M}^{\sim\sim}$ is equal to $\mathscr{A}$, then $\mathscr{M}$ is σ-weakly dense in $\mathscr{A}$.*

Proof. We use the same notation as Theorem 4. Every $f \in \mathscr{B}$ has the form $f = \sum \{f_i \cdot \phi_i \mid 1 \leqq i \leqq n\}$ where $f_i \in \mathscr{Z}_*$ and $\phi_i \in \mathscr{M}^\sim$. Because every f_i may be written as a linear combination of positive functionals in $\mathscr{Z}_*$, we may assume every f_i is positive. Then $\sum f_i = w_x$ for some $x \in H$. By the Radon-Nikodym theorem there is a $C_i \in \mathscr{Z}^+$ such that $f_i = w_{C_i x}$ $(1 \leqq i \leqq n)$. Hence we may assume that f has the form $f = w_x \cdot \phi$ for some $\phi \in \mathscr{M}^\sim$.

We now obtain a contradiction by assuming that the complement of the weak closure $\mathscr{N}$ of the unit sphere of $\mathscr{M}$ with respect to the unit sphere $\mathscr{A}_1$ of $\mathscr{A}$ contains an element B. Standard separation theorems [12, 14.2] give an $f \in \mathscr{A}_*$ such that $\text{lub}\{\text{Re} f(A) \mid A \in \mathscr{N}\} < \text{Re} f(B)$. Here Re α denotes the real part of α. Because $\mathscr{B}$ is uniformly dense in $\mathscr{A}_*$ and because $\mathscr{N} \subset \mathscr{A}_1$, there is by the preceding paragraph an $x \in H$ and a $\phi \in \mathscr{M}^\sim$ such that $\text{lub}\{\text{Re}\, w_x \cdot \phi(A) \mid A \in \mathscr{N}\} < \text{Re}\, w_x \cdot \phi(B)$. We show that $\{\text{Re}\,\phi(A) \mid A \in \mathscr{N}\} = X$ is monotonely increasing in $\mathscr{Z}$. Here $\text{Re}\,\phi(A) = (\phi(A) + \phi(A)^*)/2$. For A_i $(i = 1, 2)$ in $\mathscr{N}$ let $U_i^* |\phi(A_i)| = \phi(A_i)$ be the polar decomposition of $\phi(A_i)$ in $\mathscr{Z}$. The element $\phi(U_i A_i)$ is in X because $U_i A_i \in \mathscr{N}$. Note that $\text{Re}\,\phi(A_i) \leqq |\phi(A_i)| = \phi(U_i A_i)$. There is a central projection P such that $P\phi(U_1 A_1) \leqq P\phi(U_2 A_2)$ and $(1-P)\phi(U_2 A_2) \leqq (1-P)\phi(U_1 A_1)$ and so the element $A = PU_2 A_2 + (1-P)U_1 A_1$ in $\mathscr{N}$ satisfies the relation $\text{Re}\,\phi(A_i) \leqq \phi(A)$ for $i = 1, 2$. This shows X is monotonely increasing. Because X is bounded above in $\mathscr{Z}$, it is possible to find a net $\{A_n\}$ in $\mathscr{N}$ such that $\{\text{Re}\,\phi(A_n)\}$ converges strongly to the least upper bound C_0 of X. Because $\mathscr{N}$ is compact in the weak topology, there is a subnet $\{A_{n_j}\}$ which converges weakly to an element $A_0 \in \mathscr{N}$. However $\text{Re}\,\phi(A_0) = C_0$ due to the weak continuity of ϕ (Theorem 4). By multiplying $\phi(A_0)$ by an appropriate partial isometry in $\mathscr{Z}$ we may assume that $\phi(A_0) = C_0$. For every projection P in $\mathscr{Z}$ we have $\|P\phi\| = \|PC_0\|$. On the contrary, if $\|PC_0\| < \|P\phi\|$, then $Q(C_0 + \varepsilon) \leqq (\|PC_0\| + \varepsilon)Q \leqq Q\phi(A)$ for suitable $A \in \mathscr{M} \cap \mathscr{A}_1$, $\varepsilon > 0$ and central projection Q with $0 < Q \leqq P$ because $\|P\phi\| = \text{lub}\{\|P\phi(C)\| \mid C \in \mathscr{M} \cap \mathscr{A}_1\}$. This contradicts the definition of C_0. Therefore $\|P\phi\| = \|PC_0\|$ for every central projection P. Now if $U^*|\phi(B)| = \phi(B)$ is the polar decomposition of $\phi(B)$, then $\text{Re}\, w_x(\phi(B)) = w_x(\text{Re}\,\phi(B)) \leqq w_x(|\phi(B)|) = w_x(\phi(UB))$. There is no loss of generality in assuming $\phi(B) \in \mathscr{Z}^+$. We see that $\phi(B) \leqq C_0$; otherwise there is a projection $P \neq 0$ in $\mathscr{Z}$ and an $\varepsilon > 0$ such that $P(C_0 + \varepsilon) \leqq P\phi(B)$. This means that $\|PC_0\| < \|P\phi\|$ and that is impossible. However $w_x \cdot \phi(B) \leqq w_x(C_0) = w_x(\phi(A_0))$ is obtained. Because $A_0 \in \mathscr{N}$ we have reached a contradiction. Therefore $\mathscr{N} = \mathscr{A}_1$. Q.E.D.

Corollary. *The module $\mathscr{M}^\sim$ is equal to the module $\mathscr{A}_\sim$ of all σ-weakly continuous functionals of the module $\mathscr{A}$.*

Proof. If $\phi \in \mathscr{A}_\sim$, then the restriction $\phi|\mathscr{M}$ of ϕ to $\mathscr{M}$ defines a bounded functional of the module $\mathscr{M}$. There is a $\psi \in \mathscr{M}^\sim$ such that $\psi = \phi|\mathscr{M}$. Because $\psi \in \mathscr{A}_\sim$ (Theorem 4) and because $\mathscr{M}$ is σ-weakly dense in $\mathscr{A}$, we have that $\psi = \phi$. So $\mathscr{A}_\sim \subset \mathscr{M}^\sim$. Since $\mathscr{M}^\sim \subset \mathscr{A}_\sim$ (Theorem 4), we have $\mathscr{A}_\sim = \mathscr{M}^\sim$. Q.E.D.

Let $\mathscr{M}$ be a Banach module over the commutative von Neumann algebra $\mathscr{Z}$;

the topology induced on the dual $\mathcal{M}^{\sim}$ of the module $\mathcal{M}$ by the family of seminorms $p_{f,A}(\phi)=|f(\phi(A))|$ ($f\in\mathcal{Z}_{*}$, $A\in\mathcal{M}$) will be called the weak topology of $\mathcal{M}^{\sim}$.

THEOREM 6. *Let $\mathcal{M}$ be a Banach module over the commutative σ-finite von Neumann algebra $\mathcal{Z}$. Then the unit sphere $\mathcal{M}_1^{\sim}$ of $\mathcal{M}^{\sim}$ is compact in the weak topology.*

Proof. Let x be a separating unit vector for $\mathcal{Z}$. Let $\{\phi_n\}$ be a net in $\mathcal{M}_1^{\sim}$. A subnet $\{w_x\cdot\phi_n \mid n\in S\}$ of $\{w_x\cdot\phi_n\}$ converges in the $\sigma(\mathcal{M}^*, \mathcal{M})$-topology of the dual $\mathcal{M}^*$ of $\mathcal{M}$ to the functional f in the unit sphere of $\mathcal{M}^*$. Because $|f(CD^*B)|\leqq\|B\|\,\|Cx\|\,\|Dx\|$ for each $B\in\mathcal{M}$ and $C, D\in\mathcal{Z}$, the relation $\langle Cx, Dx\rangle_B=f(CD^*B)$ defines a continuous bilinear functional on a dense linear manifold of the subspace

$$K = \text{closure}\,\{Cx \mid C\in\mathcal{Z}\}.$$

This bilinear functional may be uniquely extended to a bilinear functional $\langle y, z\rangle_B$ of K such that $|\langle y, z\rangle_B|\leqq\|B\|\,\|y\|\,\|z\|$ for every y and z. There is a unique bounded linear operator $A'=A'_B$ on K such that $(A'y, z)=\langle y, z\rangle_B$ for all y and z. The projection E of the Hilbert space onto K is an abelian projection of the commutator $\mathcal{Z}'$ of $\mathcal{Z}$. If $A\in\mathcal{Z}$, then $(A'AECx, Dx)=(AEA'Cx, Dx)$ for every C and D in $\mathcal{Z}$. Thus $(AE)A'=A'(AE)$, and because $A\in\mathcal{Z}$ is arbitrary, we find that A' is an element of the commutator $E\mathcal{Z}'E$ of $\mathcal{Z}E$ on K. There is unique $\phi(B)$ in $\mathcal{Z}$, because the central support of E is 1, such that $\phi(B)E=A'_B$. By the uniqueness of $\phi(B)$ we may conclude that $B\to\phi(B)$ is a module homomorphism of $\mathcal{M}$ into $\mathcal{Z}$ and by the relation $|(\phi(B)y, z)|\leqq\|B\|\,\|y\|\,\|z\|$ for all y and z in K we may conclude that $\|\phi(B)\|=\|\phi(B)E\|\leqq\|B\|$. So $\phi\in\mathcal{M}_1^{\sim}$. From the fact that $\{\|\phi_n(B)\| \mid n\in S\}$ is bounded for fixed $B\in\mathcal{M}$ and from the fact that $\{Cx \mid C\in\mathcal{Z}\}$ is dense in K, we see that $\{\phi_n(B)E \mid n\in S\}$ converges weakly to $\phi(B)E$. This implies that $\{\phi_n(B) \mid n\in S\}$ converges weakly to $\phi(B)$ for every $B\in\mathcal{M}$ [2, I, §4, Theorem 2]. Q.E.D.

A subset $\mathcal{K}$ of a module $\mathcal{M}$ over the commutative von Neumann algebra $\mathcal{Z}$ is said to be $\mathcal{Z}$-convex if $CA+(1-C)B$ is in $\mathcal{K}$ for every A and B in $\mathcal{K}$ and C in $\mathcal{Z}$ with $0\leqq C\leqq 1$. A nonvoid $\mathcal{Z}$-convex subset $\mathcal{K}'$ of a $\mathcal{Z}$-convex subset $\mathcal{K}$ of $\mathcal{M}$ is said to be a support of $\mathcal{K}$ if $CA+(1-C)B\in\mathcal{K}'$ for A and B in $\mathcal{K}$ and C in $\mathcal{Z}$ with $0<C<1$ implies $A\in\mathcal{K}'$ and $B\in\mathcal{K}'$. By $0<C<1$ it is meant that C is strictly between 0 and 1, i.e. the spectrum of C is contained in the open interval (0, 1). A point A of a $\mathcal{Z}$-convex subset $\mathcal{K}$ of $\mathcal{M}$ is said to be an extreme point of $\mathcal{K}$ if $\{A\}$ is a support of $\mathcal{K}$.

The following is a Krein-Milman Theorem for $\mathcal{M}^{\sim}$.

THEOREM 7. *Let $\mathcal{M}$ be a Banach $\mathcal{Z}$-module whose second dual $\mathcal{M}^{\sim\sim}$ is a von Neumann algebra $\mathcal{A}$ with center $\mathcal{Z}$. Let P be a nonzero projection in $\mathcal{Z}$. Then there is an extreme point ϕ in the unit sphere $\mathcal{S}$ of positive functionals of $\mathcal{A}_{\sim}$ such that $P\phi\neq 0$.*

Proof. We may assume that P is σ-finite since every nonzero projection in $\mathcal{Z}$ majorizes a nonzero σ-finite projection. Let $\mathcal{H}=\{\text{Re}\,\phi|\phi\in\mathcal{A}_{\sim}$ and $P\phi=\phi\}$. Here

$\mathrm{Re}\,\phi(A)=(\phi(A)+\phi(A)^*)/2$. The set $\mathscr{H}$ has a natural module structure over the ring of hermitian elements in $\mathscr{Z}$. The seminorms $p_{f,A}$ ($f\in\mathscr{Z}_*$, $A\in\mathscr{M}$) induce a weak topology on H in which the $\mathscr{Z}$-convex subset $\mathscr{H}_1=\{\mathrm{Re}\,\phi \mid \phi\in\mathscr{A}_\sim, P\phi=\phi, \|\phi\|\leqq 1\}$ is compact since the unit sphere of $\mathscr{M}^\sim$ is compact in the weak topology and since the adjoint operation of $\mathscr{Z}$ is weakly continuous. Suppose that $\mathscr{H}_1$ has a nonzero extreme point ϕ'. We show that $\mathscr{S}$ has a nonzero extreme point ψ such that $P\psi=\psi$. Let ϕ be an element of the unit sphere of $\mathscr{A}_\sim$ such that $P\phi=\phi$ and $\mathrm{Re}\,\phi=\phi'$. If $C\theta+(1-C)\theta'=\phi$ for θ, θ' in the unit sphere of $\mathscr{A}_\sim$ and C strictly between 0 and 1 in $\mathscr{Z}$, then $C\,\mathrm{Re}\,\theta+(1-C)\,\mathrm{Re}\,\theta'=\phi'$ and consequently $\phi'=\mathrm{Re}\,\theta=\mathrm{Re}\,\theta'$. This means that $\phi=\theta=\theta'$ and thus that ϕ is an extreme point of the unit sphere of $\mathscr{A}_\sim$. Using the polar decomposition for ϕ, we obtain a partial isometric operator U in $\mathscr{A}P$ such that $\psi(A)=\phi(UA)$ defines a functional in $\mathscr{S}$ and such that $\psi(U^*A)=\phi(A)$ for every A in $\mathscr{A}$. If $\psi=C\theta+(1-C)\theta'$ for $\theta,\theta'\in\mathscr{S}$ and $C\in\mathscr{Z}$ with $0<C<1$, then $\phi(A)=\theta(U^*A)=\theta'(U^*A)$ for every $A\in\mathscr{A}$. But $\psi(U^*U)=\psi(1)$ and so both θ and θ' vanish on the projection $1-U^*U$. By the Cauchy-Schwarz inequality we have that $\theta(A)=\theta(U^*UA)$ and $\theta'(A)=\theta'(U^*UA)$ for every $A\in\mathscr{A}$. We then see that $\psi=\theta=\theta'$ and thus that ψ is a nonzero extreme point of $\mathscr{S}$ such that $P\psi=\psi$. We complete the proof by showing that $\mathscr{H}_1$ has a nonzero extreme point.

Now let $\mathscr{F}$ be a maximal family with the finite intersection property of compact supports of $\mathscr{H}_1$. The set $\mathscr{H}_0=\bigcap\mathscr{F}$ is nonvoid and evidently a compact support of $\mathscr{H}_1$. We show that $\mathscr{H}_0$ contains a single point. On the contrary if ϕ and ψ are two distinct points in $\mathscr{H}_0$, there is an element A in $\mathscr{M}$ such that $\phi(A)\neq\psi(A)$. Indeed, an element in $\mathscr{A}_\sim$ which vanishes under multiplication by $1-P$ is uniquely determined by its values on the elements of $\mathscr{M}$. The set $\{\theta(A)\mid\theta\in\mathscr{H}_0\}$ is a monotonely increasing net in set of hermitian elements of $\mathscr{Z}$ and therefore the set

$$\mathscr{H}_0(A_0) = \{\theta\in\mathscr{H}_0 \mid \theta(A) = A_0 = \mathrm{lub}\,\{\theta'(A)\mid\theta'\in\mathscr{H}_0\}\}$$

is nonvoid. This is an obvious variation of the statement concerning the set X in Theorem 5. The set $\mathscr{H}_0(A_0)$ is certainly $\mathscr{Z}$-convex and weakly compact. It is a support of $\mathscr{H}_0$ for $C\theta(A)+(1-C)\theta'(A)=A_0$ for θ, θ' in $\mathscr{H}_0$ and C in $\mathscr{Z}$ with $0<C<1$ implies $\theta(A)=\theta'(A)=A_0$. However the compact support $\mathscr{H}_0(A_0)$ of $\mathscr{H}_0$ is a compact support of $\mathscr{H}_1$ which cannot contain both ϕ and ψ. This contradicts the maximality of $\mathscr{F}$. Hence we must conclude that $\mathscr{H}_0$ contains a single point ϕ_0. We have that $\phi_0\neq 0$. On the contrary, if E is an abelian projection in the commutator of $\mathscr{Z}$ which has central support P, then $\theta=\mathrm{Re}\,\tau_E\in\mathscr{H}_1$ (Corollary, Theorem 5) and $(\theta-\theta)/2=0$. But $\theta=-\theta=0$ is evidently not true. So $\phi_0\neq 0$. Q.E.D.

Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{Z}$; a positive functional $\phi\in\mathscr{A}^\sim$ is said to be $\mathscr{Z}$-irreducible on $\mathscr{A}$ if given a positive functional ψ in $\mathscr{A}^\sim$ with $\psi\leqq\phi$ then there is a $C\in\mathscr{Z}^+$ such that $C\phi=\psi$.

THEOREM 8. *Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{Z}$. For a functional ϕ*

in the unit sphere $\mathscr{S}$ of the set of positive functionals of $\mathscr{A}^{\sim}$ the following are equivalent: (1) ϕ *is an extreme point of* $\mathscr{S}$; *and* (2) $\phi(1)$ *is a projection of* $\mathscr{Z}$ *and* ϕ *is* $\mathscr{Z}$-*irreducible.*

Proof. Let ϕ be an extreme point of $\mathscr{S}$. Let P be the support of the positive element $\phi(1)$ in the unit sphere of $\mathscr{Z}$. If there is a ζ in the spectrum Z of $\mathscr{Z}$ such that $0<\phi(1)^\wedge(\zeta)<1$, there is an $\varepsilon>0$ and a projection Q majorized by P with $Q^\wedge(\zeta)=1$ and $\varepsilon Q\leqq Q\phi(1)\leqq(1-\varepsilon)Q$. So $\|Q\phi\|\leqq 1-\varepsilon$ and $\theta=(1-\varepsilon)^{-1}$. $Q\phi+(1-Q)\phi$ is a point of $\mathscr{S}$. However $\phi=(1-\varepsilon)\theta+\varepsilon(1-Q)\phi$. This gives a contradiction. So we conclude that $\phi(1)=P$. Now let $\psi\in\mathscr{S}$ and $\psi\leqq\phi$. Setting $\theta=\phi-\psi$ we assume that $\theta(1)^\wedge(\zeta)\neq 0$ and $\theta(1)^\wedge(\zeta)\neq 1$ for some $\zeta\in Z$. There is an $\varepsilon>0$ and a central projection Q such that $Q^\wedge(\zeta)=1$ and $\varepsilon Q\leqq Q\theta(1)\leqq(1-\varepsilon)Q$. This means that $B\theta(1)=Q$ and $D\psi(1)=Q$ for suitable B and D in $(\mathscr{Z}Q)^+$. Setting $C=((1-Q)/2)+Q\theta(1)$, we obtain a central element strictly between 0 and 1 satisfying the relation

$$\phi = C(B\theta+(1-Q)\phi)+(1-C)(D\psi+(1-Q)\phi).$$

Because $B\theta+(1-Q)\phi$ and $D\psi+(1-Q)\phi$ are in $\mathscr{S}$, $D\psi=Q\phi$ and so $Q\psi=Q\psi(1)\phi$. If $\{Q_n\}$ is a maximal set of mutually orthogonal nonzero central projections such that $Q_n\psi(1)\phi=Q_n\psi$ for each Q_n, then $\sum Q_n=P$. This proves that $\psi=P\psi=\psi(1)\phi$. Thus, we see that ϕ is $\mathscr{Z}$-irreducible.

Conversely, let ϕ be $\mathscr{Z}$-irreducible and let $\phi(1)$ be a projection P. If $\phi=C\psi+(1-C)\theta$ for $\psi,\theta\in\mathscr{S}$ and $0<C<1$ in $\mathscr{Z}$, then there is a $D\in\mathscr{Z}^+$ such that $C\psi=D\phi$. Then the two relations $C^{-1}DP=\psi(1)$ and $P=C\psi(1)+(1-C)\theta(1)$ imply $P=\psi(1)$. So $\psi=C^{-1}D\phi=\phi$. Similarly $\theta=\phi$. This proves ϕ is an extreme point of $\mathscr{S}$. Q.E.D.

THEOREM 9. *Let $\mathscr{Z}$ be a commutative von Neumann algebra and let $\mathscr{A}$ be a C^*-algebra with center $\mathscr{Z}$. Then $\mathscr{A}$ is a type* I *von Neumann algebra if and only if the module $\mathscr{A}$ is the second dual $\mathscr{M}^{\sim\sim}$ of a Banach $\mathscr{Z}$-module $\mathscr{M}$.*

Proof. If $\mathscr{A}$ is a type I von Neumann algebra, then $\mathscr{A}$ is the second dual of the closed two-sided I_a of $\mathscr{A}$ generated by the abelian projections of $\mathscr{A}$ [9, §4].

Conversely, it is already known that $\mathscr{A}$ is a von Neumann algebra (Theorem 4) and so it is sufficient to prove that every nonzero projection P of $\mathscr{Z}$ majorizes a nonzero abelian projection. Let ϕ be an extreme point of the set of positive functionals $\mathscr{S}$ of the unit sphere of $\mathscr{A}_\sim$ such that $P\phi\neq 0$. Then it is easy to see that $P\phi$ is an extreme point of $\mathscr{S}$ and so we assume that $P\phi=\phi$. Let x be a unit vector in the Hilbert space H of $\mathscr{A}$ such that $\phi(1)x=x$ (Theorem 8). Let g be a positive functional on $\mathscr{A}$ majorized by $w_x\cdot\phi$. If $A\in\mathscr{Z}^+$, then $g(A)\leqq w_x(\phi(1)A)=w_x(A)$. There is an element C in $\mathscr{Z}^+$ such that $g(A)=(ACx, Cx)$ for all A in $\mathscr{Z}$. By Theorem 1 there is a normal functional ψ of the module $\mathscr{A}$ such that $g=w_{Cx}\cdot\psi$ because g is normal. If E is the projection of H onto closure $\{Ax \mid A\in\mathscr{Z}\}$ then $C^2\psi(A)E\leqq\phi(A)E$ for every A in $\mathscr{A}^+$ since $(C^2\psi(A)Ey, Ey)\leqq(\phi(A)Ey, Ey)$ for every y in $E(H)$. However the projection E is in the commutator of $\mathscr{Z}$ on H and so if Q is the central support of E we have that $C^2Q\psi(A) \leqq Q\phi(A) \leqq \phi(A)$ for every A in $\mathscr{A}^+$. There is a $D \in \mathscr{Z}^+$ such that $C^2Q\psi = D\phi$ (Theorem 8). Thus $g(A) = w_x \cdot \phi(DA)$ for all A in $\mathscr{A}$. This means that the projection in $\mathscr{A}$ corresponding to closure $\{A'x \mid A' \in \mathscr{A}'\}$ where $\mathscr{A}'$ is the commutator of $\mathscr{A}$ is abelian in $\mathscr{A}$ [8, §2]. This completes the proof. Q.E.D.

REMARK. Actually we have proved that a normal functional f on $\mathscr{A}$ is centrally reducible (cf. [8, §2]) if and only if there is a $\mathscr{Z}$-irreducible normal functional ϕ of the module $\mathscr{A}$ such that $f \cdot \phi = f$.

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